Asymptotic spectrum of the oblate spin-weighted spheroidal harmonics: a WKB analysis

Shahar Hod
The Ruppin Academic Center, Emeq Hefer 40250, Israel
and
The Hadassah Institute, Jerusalem 91010, Israel
(Dated: April 3, 2013)

Spin-weighted spheroidal harmonics play a central role in the mathematical description of diverse physical phenomena, including black-hole perturbation theory and wave scattering. We present a novel and compact derivation of the asymptotic eigenvalues of these important functions. Our analysis is based on a simple trick which transforms the corresponding spin-weighted spheroidal angular equation into a Schrödinger-like wave equation which is amenable to a standard WKB analysis.

Spin-weighted spheroidal harmonics $S(\theta;c)$ have attracted much attention over the years from both physicists and mathematicians. These special functions are solutions of the angular differential equation [1–3]

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial S}{\partial\theta}\Big)+\Big[c^2\cos^2\theta-2cs\cos\theta-\frac{(m+s\cos\theta)^2}{\sin^2\theta}+s+A\Big]S=0\ , \tag{1}$$

where $\theta\in[0,\pi]$ and $c\in\mathbb{Z}$. When $c\in\mathbb{R}$, the case we shall study in this Letter, the eigenfunctions are called oblate. The parameters m and s are the azimuthal harmonic index and the spin-weight of the wave field, respectively [2]. These quantum numbers can assume integer or half-integer values [2]. When $s=0$ the spin-weighted spheroidal harmonics reduce to the familiar scalar spheroidal harmonics [1–3] which play a central role in the mathematical description of diverse physical phenomena such as electromagnetic wave scattering [4], the quantum-mechanical description of the hydrogen molecular ion H_2^+ [5], and modern models of nuclear structure [6].

The general (non-zero) spin case was first studied by Teukolsky [2, 7, 8] in the context of black-hole perturbation theory [9]; in this case the parameter s denotes the type of the perturbation-field: $s=0$ for scalar perturbations, $s=\pm\frac{1}{2}$ for massless neutrino perturbations, $s=\pm1$ for electromagnetic perturbations, and $s=\pm2$ for gravitational perturbations.

The angular functions $S(\theta;c)$ are required to be regular at the poles $\theta=0$ and $\theta=\pi$. These boundary conditions pick out a *discrete* set of eigenvalues $\{{}_sA_{lm}\}$ labeled by the discrete parameter l. In the $c\to0$ limit the angular functions become the familiar spin-weighted spherical harmonics with the well-known angular eigenvalues ${}_sA_{lm}=l(l{+}1)-s(s{+}1)$ [with $l\geq\max(|m|,|s|)$]. The opposite limit, $c\to\infty$ (with fixed m), was studied in [1, 10] for the $s=0$ case and in [11–13] for the general spin case. It was found that the asymptotic eigenvalues are given by [12, 14]:

$${}_sA_{lm}=-c^2+\beta|c|+O(1)\ , \tag{2}$$

where the function β depends on the spin-parameter s, the azimuthal harmonic index m, and the spheroidal harmonic index l [see Eqs. (16)-(17) below].

While correct, the derivation of (2) presented in [12] for the general spin case is rather complex and lengthy. The aim of the present Letter is to present an alternative and (much) shorter derivation of the formula (2) which, we believe, is also very simple and intuitive. The trick is to transform the angular equation (1) into the form of a Schrödinger-like wave equation and then to perform a standard WKB analysis. It proves useful to introduce the coordinate x defined by [15]

$$x\equiv\ln\Big(\tan\Big(\frac{\theta}{2}\Big)\Big)\ , \tag{3}$$

in terms of which the angular equation (1) becomes a Schrödinger-like wave equation of the form [16]

$$\frac{d^2S}{dx^2}-US=0\ , \tag{4}$$

where

$$U(\theta)=(m+s\cos\theta)^2-\sin^2\theta(c^2\cos^2\theta-2cs\cos\theta+s+A)\ . \tag{5}$$

Note that the interval $\theta \in [0,\pi]$ maps into $x \in [-\infty,\infty]$. The Schrödinger-type angular equation (4) is now in a form that is amenable to a standard WKB analysis.

The effective potential $U(\theta)$ is in the form of an asymmetric double-well potential [17]: in the $c \to \infty$ limit it has a local maximum at

$$\theta_{\text{max}} = \frac{\pi}{2} + O(sc^{-1}) \quad \text{with} \quad U(\theta_{\text{max}}) = c^2 - \beta c + O(1) \tag{6}$$

and two local minima at

$$\theta^-_{\text{min}} = \sqrt{\frac{\beta - 2s}{2c}} + O(c^{-3/2}) \quad \text{and} \quad \theta^+_{\text{min}} = \pi - \sqrt{\frac{\beta + 2s}{2c}} + O(c^{-3/2}) \tag{7}$$

with

$$U(\theta^{\pm}_{\text{min}}) = (m \mp s)^2 - \frac{1}{4}(\beta \pm 2s)^2 + O(c^{-1}) \ . \tag{8}$$

Thus, in the $c \to \infty$ limit the two potential wells are separated by a large potential-barrier of height

$$\Delta U \equiv U(\theta_{\text{max}}) - U(\theta^{\pm}_{\text{min}}) = c^2 + O(c) \to \infty \quad \text{as} \quad c \to \infty \ . \tag{9}$$

Regions where $U(\theta) < 0$ are characterized by an oscillatory behavior of the wave-function S (the 'classically allowed regions'), while regions with $U(\theta) > 0$ (the 'classically forbidden regions') are characterized by an exponential behavior (evanescent waves). The 'classical turning points' are characterized by $U = 0$. There are two pairs $\{\theta^-_{1,2}, \theta^+_{1,2}\}$ of such turning points (with $\theta^-_1 < \theta^-_{\text{min}} < \theta^-_2 < \theta_{\text{max}} < \theta^+_1 < \theta^+_{\text{min}} < \theta^+_2$), which in the $c \to \infty$ limit are located in the immediate vicinity of the two minima $\{\theta^-_{\text{min}}, \theta^+_{\text{min}}\}$:

$$\theta^-_{1,2} = \sqrt{\frac{\beta - 2s \pm \sqrt{(\beta-2s)^2 - 4(m+s)^2}}{2c}} + O(c^{-3/2}) \quad \text{and}$$
$$\theta^+_{1,2} = \pi - \sqrt{\frac{\beta + 2s \pm \sqrt{(\beta+2s)^2 - 4(m-s)^2}}{2c}} + O(c^{-3/2}) \ . \tag{10}$$

A standard textbook second-order WKB approximation for the bound-state 'energies' of a Schrödinger-like wave equation of the form (4) yields the well-known quantization condition [18–22]

$$\int_{x^{\pm}_1}^{x^{\pm}_2} dx \sqrt{-U(x)} = (N + \frac{1}{2})\pi \quad ; \quad N = \{0,1,2,...\} \ , \tag{11}$$

where $x^-_{1,2}$ and $x^+_{1,2}$ are the turning points [with $U(x^{\pm}_{1,2}) = 0$] of the left and right potential wells, respectively, and N is a non-negative integer. Here we have used the fact that in the $c \to \infty$ limit the two potential wells are separated by an infinite potential-barrier [see Eq. (9)]. Thus, in the $c \to \infty$ limit the coupling between the wells (the 'quantum tunneling' through the potential barrier) is negligible and the two potential wells can therefore be treated as independent of each other [18, 23, 24].

Using the relation $dx/d\theta = 1/\sin\theta$, one can write the WKB condition (11) in the form

$$\int_{\theta^{\pm}_1}^{\theta^{\pm}_2} d\theta \frac{\sqrt{-U(\theta)}}{\sin\theta} = (N + \frac{1}{2})\pi \quad ; \quad N = \{0,1,2,...\} \ . \tag{12}$$

The WKB quantization conditions (12) determine the eigenvalues $\{A\}$ of the associated spin-weighted spheroidal harmonics in the large-c limit. The relation so obtained between the eigenvalues and the parameters c, m, s and N is rather complex and involves elliptic integrals. However, given the fact that in the $c \to \infty$ limit the turning points $\theta^-_{1,2}$ and $\theta^+_{1,2}$ lie in the immediate vicinity of $\theta = 0$ and $\theta = \pi$ respectively [see Eq. (10)], one can approximate the integrals in (12) by [25]

$$\int_{\theta^-_1}^{\theta^-_2} d\theta \sqrt{(\beta - 2s)c - c^2\theta^2 - \frac{(m+s)^2}{\theta^2}} = (N + \frac{1}{2})\pi \quad \text{and} \quad \int_{\theta^+_1}^{\theta^+_2} d\theta \sqrt{(\beta + 2s)c - c^2(\pi-\theta)^2 - \frac{(m-s)^2}{(\pi-\theta)^2}} = (N + \frac{1}{2})\pi \ . \tag{13}$$

Defining $\phi \equiv \theta\sqrt{\frac{c}{\beta-2s}}$ for the left potential well and $\phi \equiv (\pi-\theta)\sqrt{\frac{c}{\beta+2s}}$ for the right well, one can write the two quantization conditions (13) in a unified and compact form

$$(\beta \mp 2s)\int_{\phi_1^\pm}^{\phi_2^\pm} d\phi\sqrt{1-\phi^2-\frac{b_\pm^2}{\phi^2}} = (N+\frac{1}{2})\pi\ , \tag{14}$$

where $b_\pm \equiv |m\pm s|/(\beta\mp 2s)$ and $\{\phi^-_{1,2},\phi^+_{1,2}\}$ are the rescaled turning points [where the integrands of (14) vanish]. Evaluating the integrals in (14) is straightforward, and one finds

$$\beta^\pm(N) = 2(2N+1\pm s+|m\pm s|)\quad ;\quad N=\{0,1,2,...\} \tag{15}$$

for the two quantized spectra $\{\beta^-(N),\beta^+(N)\}$ (which correspond to the two potential wells).

Note that $|\beta^+(N)-\beta^-(N)|/4 = |s+\frac{1}{2}(|m+s|-|m-s|)|$ is always an integer, which implies that the two spectra (15) are doubly degenerate above some eigenvalue. Thus, defining

$$\beta_0^\pm \equiv 2(1\pm s+|m\pm s|)\ , \tag{16}$$

we can unify the two spectra (15) and write them in the form of a single compact formula:

$$\beta(N) = 4N+\min(\beta_0^-,\beta_0^+)\quad ;\quad N=\{0,1,2,...\} \tag{17}$$

where the spectrum (17) is doubly degenerate for $N \geq |\beta_0^+-\beta_0^-|/4$ [26].

ACKNOWLEDGMENTS

This research is supported by the Carmel Science Foundation. I thank Uri Keshet, Oded Hod, Yael Oren, Arbel M. Ongo and Ayelet B. Lata for helpful discussions.

[1] C. Flammer, *Spheroidal Wave Functions* (Stanford University Press, Stanford, 1957).
[2] S. A. Teukolsky, Phys. Rev. Lett. **29**, 1114 (1972); Astrophys. J. **185**, 635 (1973).
[3] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, New York, 1970).
[4] S. Asano and G. Yamamoto, Appl. Opt. **14**, 29 (1975); N. V. Voshchinnikov and V. G. Farafonov, Astrophys. Space. Sci. **204**, 19 (1993); M. I. Mishchenko, J. W. Hovenier and L. D. Travis, Eds., *Light Scattering by Nonspherical Particles: Theory, Measurements, and Applications* (Academic Press, New York, 2000).
[5] H. Eyring, J. Walter and G.E. Kimball, *Quantum Chemistry* (Wiley, New York, 1948); M. M. Madsen and J. M. Peek, *Eigenparameters for the lowest twenty electronic states of the hydrogen molecule ion*, Atomic Data **2**, 171 (1971).
[6] B. D. B. Figueiredo, J. Phys. A. **35**, 2877 (2002).
[7] E. Berti, V. Cardoso and A. O. Starinets, Class. Quant. Grav. **26**, 163001 (2009).
[8] S. Hod, Phys. Rev. Lett. **84**, 10 (2000) [arXiv:gr-qc/9907096]; U. Keshet and S. Hod, Phys. Rev. D (Rapid communication) **76**, 061501 (2007) [arXiv:0705.1179]; U. Keshet, arXiv:1207.2460.
[9] In the context of black-hole perturbation theory, the parameter c stands for $a\omega$, where a is the angular momentum per unit mass of the black hole and ω is the (conserved) frequency of the perturbation mode [2].
[10] J. Meixner and F. W. Schäfke, *Mathieusche Funktionen und Sphäroidfunktionen mit Anwendungen auf Physikalische und Technische Probleme* (Springer-Verlag, Berlin, 1954).
[11] R. A. Breuer, *Gravitational Perturbation Theory and Synchrotron Radiation* (Lecture Notes in Physics, Vol. 44), (Springer, Berlin 1975); R. A. Breuer, M. P. Ryan Jr, and S. Waller, Proc. R. Soc. London **A358**, 71 (1977).
[12] M. Casals and A. C. Ottewill, Phys. Rev. D **71**, 064025 (2005).
[13] E. Berti, V. Cardoso, and M. Casals, Phys. Rev. D **73**, 024013 (2006); Erratum-ibid. D **73**, 109902 (2006).
[14] We shall assume without loss of generality that $c>0$. For $c<0$ one should simply replace $\{s,m\}\to\{-s,-m\}$ in the final answer [see Eqs. (16)-(17) below].
[15] H. Yang, D. A. Nichols, F. Zhang, A. Zimmerman, Z. Zhang and Y. Chen, arXiv:1207.4253.
[16] Note that in the quantum-mechanical terminology $-U$ stands for $\frac{2m}{\hbar^2}(E-V)$, where E,V, and m are the total energy, potential energy, and mass of the particle, respectively.
[17] The effective potential $U(\theta)$ becomes symmetric (invariant under the transformation $\theta\to\pi-\theta$) in the scalar ($s=0$) case.
[18] L. D. Landau and E. M. Liftshitz, *Quantum Mechanics*, 3rd ed. (Pergamon, New York, 1977), Chap. VII.
[19] J. Heading, *An Introduction to Phase Integral Methods* (Wiley, New York, 1962).
[20] C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (McGraw-Hill, New York, 1978), Chap. 10.

[21] B. F. Schutz and C. M. Will, Astrophys. J. **291**, L33 (1985); S. Iyer and C. M. Will, Phys. Rev. D **35**, 3621 (1987).

[22] Higher order corrections to the asymptotic eigenvalues [see formula (17) below] can be obtained by using a higher-order WKB analysis [21].

[23] C. S. Park, M. G. Jeong, S. K. Yoo, and D. K. Park, Phys. Rev. A, **58**, 3443 (1998); Z. Cao, Q. Liu, Q. Shen, X. Dou, Y. Chen, and Y. Ozaki, Phys. Rev. A **63**, 054103 (2001); F. Zhou, Z. Cao, and Q. Shen, Phys. Rev. A **67**, 062112 (2003).

[24] More precisely, the coupling between the two wells (due to the weak 'quantum tunneling' through the high potential barrier) introduces a correction term of order $\exp[-\int_{\theta_2^-}^{\theta_1^+} d\theta\sqrt{U(\theta)}]$ to the r.h.s of the WKB quantization condition (11) [18, 23]. This term is of the order of $e^{-\sqrt{\Delta U}} \sim e^{-c} \to 0$ [see Eq. (9)] and is therefore negligible in the $c \to \infty$ limit [18, 23].

[25] Here we have used the relations $\cos\theta = 1-\frac{1}{2}\theta^2+O(\theta^4)$ and $\sin\theta = \theta+O(\theta^3)$ for $\theta \simeq 0$ and $\cos\theta = -1+\frac{1}{2}(\pi-\theta)^2+O((\pi-\theta)^4)$ and $\sin\theta = \pi-\theta+O((\pi-\theta)^3)$ for $\theta \simeq \pi$.

[26] Note that our discrete resonance parameter N corresponds to $l - l_{\min}$ in the non-degenerate regime $0 \le N < |\beta_0^+ - \beta_0^-|/4$ and to the integral part of $\frac{1}{2}(l - l_{\min}) + |\beta_0^+ - \beta_0^-|/8$ in the degenerate regime $N \ge |\beta_0^+ - \beta_0^-|/4$. Here l is the spheroidal harmonic index [12] with $l_{\min} \equiv \max(|m|, |s|)$.